IX. *Occultatio* Jovis *à* Luna, *obſ.* Londini.

Read June 7. 1744.

Notante Horologio.

	d	h	′	″	
1744. *June*	6.	11	13	40	Immerſio centri *Jovialis*, ſat certe.
			35	14	α Serpentaria culminat.
			43	15	Emerſio centri, raptim inter nubes.

Tubo 12 Ped.

J. Bevis.

X. *A letter from Mr.* Abraham De Moivre, *F. R. S. to* William Jones, *Eſquire, F. R. S. concerning the eaſieſt method for calculating the value of annuities upon lives, from tables of obſervations.*

S I R,

Preſented June 7. 1744.

YOU may remember, that ſome time after the printing of the ſecond edition of my book of annuities on lives, you told me, that it ſeem'd ſtrange to you, that, conſidering I had demonſtrated the chief propoſitions in the book, I had neglected to demonſtrate the theorem, which is found in page 86. line 12. which, you ſaid, of all the reſt, appear'd to you the moſt curious; I anſwer'd, that, as the demonſtration depended upon a principle which was not commonly

I known

known, I was afraid that the publiſhing of it would have ſwell'd the book too much; for this reaſon eſpecially, that many corollaries were annexed to it. However, I promiſed to ſend it you in a ſhort time; but deſired you to let me know, whether you thought it deſerv'd a place in the Philoſophical Tranſactions. I now diſcharge my promiſe; and expect, with impatience, the favour of your opinion. I am, with a very particular regard,

SIR,

Your moſt humble,

and moſt obedient Servant,

A. De Moivre.

A ſhort method of calculating the value of annuities on lives, from tables of obſervations.

ALTHO' it has been an eſtabliſhed cuſtom, in the payment of annuities on lives, that the laſt rent is loſt to the heirs of the late poſſeſſor of an annuity, if the perſon happens to die before the expiration of the term agreed on for payment, whether yearly, half-yearly, or quarterly: nevertheleſs, in this treatiſe I have ſuppos'd, that ſuch a part of the rent ſhould be paid to the heirs of the late poſſeſſor, as may be exactly proportion'd to the time elaps'd between that of the laſt payment, and the very

very moment of the life's expiring; and this by a proper, accurate, and geometrical calculation.

I have been induced to take this method, for the following reasons; first, by this supposition, the value of lives would receive but an inconsiderable increase; secondly, by this means, the several intervals of life, which, in the tables of observations, are found to have uniform decrements, may be the better connected together. It is with this view that I have framed the two following problems, with their solutions.

Problem I.

To find the value of an annuity, so circumstantiated, that it shall be on a life of a given age; and that, upon the failing of that life, such a part of the rent shall be paid to the heirs of the late possessor of an annuity, as may be exactly proportioned to the time intercepted between that of the last payment, and the very moment of the life's failing.

Solution.

LET n represent the complement of life, that is, the interval of time between the given age, and the extremity of old-age, suppos'd at 86.

r the amount of 1 *l.* for one year.

α the logarithm of r.

P the present value of an annuity of 1 *l.* for the given time.

Q the value of the life sought.

Then $\frac{1}{r-1} - \frac{P}{\alpha n} = Q.$

DEMONSTRATION.

For, let z represent any indeterminate portion of n. Now the probability of the life's attaining the end of the interval z, and then failing, is to be expressed by $\frac{\dot{z}}{n}$, (as shewn in page 77, edit. 1. and in page 115, edit. 2. of my book of annuities upon lives) upon the supposition of a perpetual and uniform decrement of life.

But it is well known, that if an annuity certain, of 1 *l.* be paid during the time z, its present value will be $P = \frac{1 - r^{\frac{1}{z}}}{r-1}$ or $\frac{1}{r-1} - \frac{1}{\overline{r-1} \times r^z}$.

And, by the laws of the doctrine of chances, the expectation of such a life, upon the precise interval z, will be expressed by $\frac{\dot{z}}{n \times \overline{r-1}} - \frac{\dot{z}}{nr^z \times \overline{r-1}}$; which may be taken for the ordinate of a curve, whose area is as the value of the life required.

In order to find the area of this curve, let $p = n \times \overline{r-1}$; and then the ordinate will become $\frac{\dot{z}}{p} - \frac{\dot{z}}{pr^z}$, a much more commodious expression.

Now it is plain, that the fluent of the first part is $\frac{z}{p}$: but as the fluent of the second part is not so readily discover'd, it will not be improper, in this place, to shew by what artifice I found it; for I do not know, whether the same method has been made use of by others: all that I can say, is, that I never had

had occasion for it, but in the particular circumstance of this problem.

Let, therefore, $r^z = x$; hence $z \log. r = \log. x$; therefore $\dot{z} \log. r =$ (fluxion of the $\log. x =$) $\frac{\dot{x}}{x}$, or $\alpha \dot{z} = \frac{\dot{x}}{x}$; consequently $\dot{z} = \frac{\dot{x}}{\alpha x}$, and $\frac{\dot{z}}{r^z} = \frac{\dot{x}}{\alpha xx}$: but the fluent of $\frac{\dot{x}}{\alpha xx}$ is $(-\frac{1}{\alpha x} =)\ -\frac{1}{\alpha r^z}$; and therefore the fluent of $-\frac{\dot{z}}{pr^z}$ will be $+\frac{1}{p\alpha r^z}$.

The sum of the two fluents will be $\frac{z}{p} + \frac{1}{p\alpha r^z}$; but, when $z = 0$, the whole fluent should be $= 0$; let therefore the whole fluent be $\frac{z}{p} + \frac{1}{p\alpha r^z} + q = 0$.

Now, when $z = 0$, then $\frac{z}{p} = 0$, and $\frac{1}{\alpha p r^z}$ becomes $\frac{1}{\alpha p}$ (for $r^z = 1$,) consequently $\frac{1}{\alpha p} + q = 0$; and $q = -\frac{1}{\alpha p}$: therefore the area of a curve, whose ordinate is $\frac{\dot{z}}{p} - \frac{\dot{z}}{pr^z}$ will be $(\frac{z}{p} - \frac{1}{\alpha p} + \frac{1}{\alpha p r^z}$ $=)\ \frac{z}{p} - \overline{1 - \frac{1}{r^z}} \times \frac{1}{\alpha p}$.

But $P = \frac{1}{r-1} - \frac{1}{\overline{r-1} \times r^z}$; therefore $1 - \frac{1}{r^z} = \overline{r-1} \times P$, and the expression for the area becomes $\frac{z}{n \times \overline{r-1}} - \frac{P}{\alpha n}$: And putting n instead of z, that area, or the value of the life, will be expressed by $\frac{1}{r-1} - \frac{P}{\alpha n}$

Q. E. D.

Those

Thoſe who are well verſed in the nature of logarithms, I mean thoſe that can deduce them from the doctrine of fluxions and infinite ſeries, will eaſily apprehend, that the quantity here called α, is that which ſome call the hyperbolic logarithm; others, the natural logarithm: it is what Mr. *Cotes* calls, the logarithm whoſe modulus is 1: laſtly, it is by ſome called *Neper*'s logarithm. And, to ſave the reader ſome trouble in the practice of this laſt theorem, the moſt neceſſary natural logarithms, to be made uſe of in the preſent diſquiſition about lives, are the following:

If $r = 1.04$, then will $\alpha = 0.0392207$.
$r = 1.05$, - - - $\alpha = 0.0487901$.
$r = 1.06$, - - - $\alpha = 0.0582589$.

It is to be obſerved, that the theorem here found, makes the values of lives a little bigger, than what the theorem found in the firſt problem of my book of annuities on lives, does; for, in the preſent caſe, there is one payment more to be made, than in the other; however, the difference is very inconſiderable.

But, altho' it be indifferent which of them is uſed, on the ſuppoſition of an equal decrement of life to the extremity of old-age; yet, if it ever happens, that we ſhould have tables of obſervations, concerning the mortality of mankind, intirely to be depended upon, then it would be convenient to divide the whole interval of life into ſuch ſmaller intervals, as, during which, the decrements of life have been obſerved to be uniform, notwithſtanding the decrements in ſome of thoſe intervals ſhould be quicker, or ſlower, than others; for then the theorem here

 found

found would be preferable to the other; as will be ſhewn hereafter.

That there are ſuch intervals, Dr. *Halley's* tables of obſervations ſufficiently ſhew; for inſtance; out of 302 perſons of 54 years of age, there remain, after 16 years (that is, of the age of 70) but 142; the decrements from year to year having been conſtantly 10; and the ſame thing happens in other intervals; and it is to be preſumed, that the like would happen in any other good tables of obſervations.

But, in order to ſhew, in ſome meaſure, the uſe of the preceding theorem, it is neceſſary to add another problem; which, tho' its ſolution is to be met with in the firſt edition of my book of annuities on lives, yet it is convenient to have it inſerted here, on account of the connexion that the application of the preceding problem has with it.

In the mean time, it will be proper to know, *What part of the yearly rent ſhould be paid to the heirs of the late poſſeſſor of an annuity, as may be exactly proportioned to the time elapſed between that of the laſt payment, and the very moment of the life's expiring.* To determine this, put A for the yearly rent; $\frac{1}{m}$ for the part of the year intercepted between the time of the laſt payment, and the inſtant of the life's failing; r the amount of 1 *l.* at the year's end: then will $\frac{r^{\frac{1}{m}}-1}{r-1}A$, be the ſum to be paid.

PRO-

PROBLEM II.

To find the value of an annuity for a limited interval of life, during which the decrements of life may be considered as equal.

SOLUTION.

LET a and b represent the number of people living in the beginning and end of the given interval of years.

s represent that interval.

P the value of an annuity certain for that interval.

Q the value of an annuity for a life supposed to be necessarily extinct in the time s; or (which is the same thing) the value of an annuity for a life, of which the complement is s.

Then $Q + \frac{b}{a} \times \overline{P-Q}$ will express the value required.

DEMONSTRATION.

For, let the whole interval between a and b be fill'd up with arithmetical mean proportionals; therefore the number of people living in the beginning and end of each year of the given interval s will be represented by the following series; *viz.*

$$a.\ \frac{sa-a+b}{s}.\ \frac{sa-2a+2b}{s}.\ \frac{sa-3a+3b}{s}.\ \frac{sa-4a+4b}{s}.\ \&c.\ \text{to}\ b.$$

Consequently, the probabilities of the life's continuing during 1, 2, 3, 4, 5, *&c.* years will be expressed by the series,

$$\frac{sa-a+b}{sa}.\ \frac{sa-2a+2b}{sa}.\ \frac{sa-3a+3b}{sa}.\ \frac{sa-4a+4b}{sa}.\ \&c.\ \text{to}\ \frac{b}{a}.$$

Wherefore,

Wherefore, the value of an annuity of 1 *l.* granted for the time *s*, will be expreſſed by the ſeries

$$\frac{sa-a+b}{sar}+\frac{sa-2a+2b}{sar^2}+\frac{sa-3a+3b}{sar^3}+\frac{sa-4a+4b}{sar^4},$$

&c. to $+\frac{b}{ar^s}$; this ſeries is diviſible into two other ſeries's, *viz.*

1*ſt.* $\frac{s-1}{sr}+\frac{s-2}{sr^2}+\frac{s-3}{sr^3}+\frac{s-4}{sr^4}$, &c. to $+\frac{s-s}{sr^s}$.

2*d.* $\frac{b}{a}\times\overline{\frac{1}{sr}+\frac{2}{sr^2}+\frac{3}{sr^3}+\frac{4}{sr^4}}$, &c. to $\frac{s}{sr^s}$.

Now, ſince the firſt of theſe ſeries's begins with a term whoſe numerator is $s-1$, and the ſubſequent numerators each decreaſe by unity; it follows, that the laſt term will be $=0$; and, conſequently, that ſeries expreſſes the value of a life neceſſarily to be extinct in the time *s*. The ſum of this ſeries may be eſteem'd as a given quantity; and is what I have expreſſed by the ſymbol *Q* in problem 1.

The ſecond ſeries is the difference between the two following ſeries's,

$$\frac{b}{a}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}+\text{\&c. to }\frac{1}{r^s}}.$$

$$\frac{b}{a}\times\overline{\frac{s-1}{sr}+\frac{s-2}{sr^2}+\frac{s-3}{sr^3}+\frac{s-4}{sr^4}\text{ \&c. to }+\frac{s-s}{sr^s}}.$$

Where, neglecting the common multiplier $\frac{b}{a}$, the firſt ſeries is the value of an annuity certain to continue *s* years; which every mathematician knows how to calculate, or is had from tables already compoſed for that purpoſe: this value is what I have called *P*; and the ſecond ſeries is *Q*.

K Therefore

Therefore $Q + \frac{b}{a} \times \overline{P-Q}$ will be the value of an annuity on a life for the limited time. *Q. E. D.*

It is obvious, that the ſeries denoted by Q, muſt of neceſſity have one term leſs than is the number of equal intervals contain'd in s; and therefore, if the whole extent of life, beginning from an age given, be divided into ſeveral intervals, each having its own particular uniform decrements, there will be, in each of theſe intervals, the defect of one payment; which to remedy, the ſeries Q muſt be calculated by problem 1.

EXAMPLE.

To find the value of an annuity for an age of 54, *to continue* 16 *years, and no longer.*

IT is found, in Dr. *Halley*'s tables of obſervations, that a is 302, and b 172: now $n = s = 16$; and, by the tables of the values of annuities certain, $P = 10.8377$; alſo (by problem 1.) $Q = (\frac{1}{r-1} - \frac{P}{an} =)$ 6.1168. Hence it follows (by this problem), that the value of an annuity for an age of 54, to continue during the limited time of 16 years, ſuppoſeing intereſt at 5 *per cent. per annum*, will be worth $(Q + \frac{b}{a} \times \overline{P-Q} =)$ 8.3365 years purchaſe.

From Dr. *Halley*'s tables of obſervations, we find, that from the age of 49 to 54 incluſive, the number of perſons, exiſting at thoſe ſeveral ages, are, 357, 346, 335, 324, 313, 302, which comprehends a ſpace of five years; and, following the precepts before laid down, we ſhall find, that an

annuity

annuity for a life of 49, to continue for the limited time of 5 years, intereſt being at 5 *per cent. per annum*, is worth 4.0374 years purchaſe.

And, in the ſame manner, we ſhall find, that the value of an annuity on life, for the limited time comprehended between the ages of 42 and 49, is worth 5.3492 years purchaſe.

Now, if it were required to determine the value of an annuity on life, to continue from the age of 42 to 70, we muſt proceed thus:

It has been proved, that an annuity on life, reaching from the age of 54 to 70, is worth 8.3365 years purchaſe; but this value, being eſtimated from the age of 49, ought to be diminiſhed on two accounts: Firſt, becauſe of the probability of the life's reaching from 49 to 54, which probability is to be deduced from the table of obſervations, and is proportional to the number of people living at the end and beginning of that interval, which, in this caſe, will be found 302 and 357: The ſecond diminution proceeds from a diſcount that ought to be made, becauſe the annuity, which reaches from 54 to 70, is eſtimated 5 years ſooner, *viz.* from the age of 49, and therefore that diminution ought to be expreſſed by $\frac{1}{r^5}$; ſo that the total diminution of the annuity of 16 years will be expreſſed by the fraction $\frac{302}{357r^5}$, which will reduce it from 8.3365 years purchaſe to 5.5259; this being added to the value of the annuity to continue from 49 to 54, *viz.* 4.0374, will give 9.5633, the value of an annuity to continue from the age of 49 to 70. For the ſame reaſon, the value 9.5633, eſtimated

 from

from the age of 42, ought to be reduced, both upon account of the probability of living from 42 to 49, and of the diſcount of money for 7 years, at 5 *per cent. per annum*, amounting together to 3.8554, which will bring it down to 5.7079; to this adding the value of an annuity on a life to continue from the age of 42 to 49, found before to be 5.3492, the ſum will be 11.0571 years purchaſe, the value of an annuity to continue from the age of 42 to 70.

In the ſame manner, for the laſt 16 years of life, reaching from 70 to 86, when properly diſcounted, and alſo diminiſhed upon the account of the probability of living from 42 to 70, the value of thoſe laſt 16 years will be reduced to 0.8; this being added to 11.0571 (the value of an annuity to continue from the age of 42 to 70, found before), the ſum will be 11.8571 years purchaſe, the value of an annuity to continue from the age of 42 to 86; that is, the value of an annuity on a life of 42; which, in my tables, is but 11.57, upon the ſuppoſition of an uniform decrement of life, from an age given to the extremity of old-age, ſuppoſed at 86.

It is to be obſerved, that the two diminutions, above-mention'd, are conformable to what I have ſaid in the corollary to the ſecond problem of the firſt edition, printed in the year 1724.

Thoſe who have ſufficient leiſure and skill to calculate the value of joint lives, whether taken two and two, or three and three, in the ſame manner as I have done the firſt problem of this tract, will be greatly aſſiſted by means of the two following theorems:

If

If the ordinate of a curve be $\frac{z}{r^2}$; its area will be $\frac{1}{a^2} - \frac{1}{a^2r^2} - \frac{z}{ar^2}$.

If the ordinate of a curve be $\frac{z^2}{r^2}$; its area will be $\frac{2}{a^3} - \frac{2}{a^3r^2} - \frac{2z}{a^2r^2} - \frac{z^2}{ar^2}$.

I beg leave, in this place, to take notice, that in the theorem in line 12. page 63. of the second edition of my book of annuities on lives, instead of *P*, it ought to be $\frac{p}{n}$*P*; where *n* and *p* represent the complements of the age, in the beginning and end of a given interval of time.

And I desire the reader of that edition to adapt the fourth article of the rule put in words at length, in page 64, to the theorem so corrected: then the solution there given, and that in page 21. of the first edition, will perfectly agree; provided that the decrements of life be supposed, in both cases, uniform, from an age given, to the extremity of old-age.

I must also take notice of an accidental error, that has crept into the 25th proposition of the second edition; which I chuse to correct as follows;

1. Let the first line of the proposition, and part of the second line, as far as *A* exclusive, be erased.

2. Let the solution proceed thus: since the life of *A* is supposed to be worth 14 years purchase, when interest is at 4 *per cent. per annum*, it follows,

from our tables, that *A* muſt be 35 years of age; therefore find, by the twenty-third propoſition, the value of an annuity of a life for 35 to continue for a limited time of 31 years: let that value be ſubducted from the value of an annuity certain, to continue 31 years; and the remainder will be the value of the reverſion.

XI. *The Appearance of a fiery* Meteor, *as ſeen by Mr.* Cradock, *communicated to the* Royal Society *by Mr.* Henry Baker, *F. R. S.*

Read June 7. 1744.

THE Head and Body emitted an extremely lucid and white Flame. The Tail appeared of a tranſparent Blue, like the Flame of Sulphur.

This *Phænomenon* was ſeen on *Sunday*, *May* 27. 1744. at 11 Minutes after 11 o' Clock at Night: Its Direction from *S. E.* to *N. W.* or thereabouts; its Height ſeemingly not half a Mile.

It was ſeen, as here deſcribed, from the Terrace in *Somerſet-Gardens*, by me,

Zach. Cradock,

Of *Somerſet-Houſe*.